Floodgate: Inference for Model-free Variable Importance

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Summary

Setup : data (Y, X, Z).

- Y : response variable ; X : the variable of interest ; $Z := (Z_1, \cdots, Z_p)$ confounders.

Q : how important each covariate (X) is in this relationship ?
A : introduce floodgate, a new inferential approach for variable importance.

- Focus on an interpretable, sensitive and nonparametric MOVI : the mMSE gap.

$$\mathcal{I}^2 = \mathbb{E}\left[(Y - \mathbb{E}[Y \mid Z])^2\right] - \mathbb{E}\left[(Y - \mathbb{E}[Y \mid X, Z])^2\right].$$

- Provide valid and robust lower confidence bounds for the mMSE gap.

$$\mu^\star := \mathbb{E}[Y \mid X, Z], \quad f(\mu) := \frac{\mathbb{E}[\mathrm{Cov}(\mu^\star(X,Z), \mu(X,Z) \mid Z)]}{\sqrt{\mathbb{E}[\mathrm{Var}(\mu(X,Z) \mid Z)]}}, \quad f(\mu) \leq \mathcal{I} \text{ for any } \mu.$$

- Allow flexible regression algorithms to obtain μ, good prediction $\Rightarrow$ good accuracy.

Genomic application to UKBB data : colored “Chicago" plot.

Zhang, Lu, and Lucas Janson. "Floodgate : inference for model-free variable importance." arXiv preprint arXiv:2007.01283 (2020).

Extensions : relax the assumption ; a different MOVI for binary responses ; group variable importance ; different covariate distribution ; adjusting for multiplicity and selection effects.

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  - Semi-parametric approaches.
  - Model-X approaches : model-X knockoffs, conditional knockoffs, conditional randomization tests, conditional permutation tests, hold-out randomization tests and so on.
  - Symmetry idea approaches : Gaussian mirrors and data splitting.

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Literature review

- Parametric approaches : Bühlmann et al. (2013), Zhang and Zhang (2014), Javanmard and Montanari (2014), Bühlmann et al. (2015), Dezeure et al. (2017), Zhang and Cheng (2017), Van de Geer et al. (2014), Nickl et al. (2013), Sur and Candès (2019), Zhao et al. (2020) ...
- Projection approaches : Buja et al. (2015, 2019a,b), Rinaldo et al. (2019), Lee et al. (2016), Taylor et al. (2014), Berk et al. (2013), Buja and Brown (2014).
- Random parameters : Lei et al. (2018), Fisher et al. (2018), Watson and Wright (2019), Rinaldo et al. (2019).
- Semi-parametric approaches : Robins et al. (2008, 2009); Li et al. (2011); Robins et al. (2017); Newey and Robins (2018), Shah and Peters (2018).
- A very recent MOVI : Azadkia and Chatterjee (2019).
- Same MOVI as us : Williamson et al. (2017).

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- (Ideally) the functional f also satisfies $f(\mu^\star) = \mathcal{I}$.

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Our choice of functional : $f(\mu) := \frac{\mathbb{E}[\mathrm{Cov}(\mu^\star(X,Z),\mu(X,Z) \mid Z)]}{\sqrt{\mathbb{E}[\mathrm{Var}(\mu(X,Z) \mid Z)]}}$

Lemma (Zhang and Janson (2020))

For any μ such that $f(\mu)$ exists, $f(\mu) \leq \mathcal{I}$ and $f(\mu^\star) = \mathcal{I}$.

Ingredients of our model-X inferential procedure :

1. $(Y_i, X_i, Z_i)_{i=1}^n$.
2. μ (can be fitted from a separate dataset e.g. sample splitting).
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- Generally, draw $\tilde{X}^{(k)}, k = 1, \cdots, K$ from $P_{X|Z}$, conditionally independently of X, Y then plug-in the Monte Carlo estimators.

Asymptotic validity

Theorem (Zhang and Janson (2020))

Under mild moment conditions on Y and $\mu(X, Z)$, we have

$$\mathbb{P}\left(L_n^\alpha(\mu) \le \mathcal{I}\right) \ge 1 - \alpha - O(n^{-1/2}).$$

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- Suggests floodgate may be robust to μ and high-dimensionality.

Statistical accuracy

1. through the best element of its equivalent class S_μ in terms of MSE

Statistical accuracy

Floodgate procedure is invariant with respect to a “equivalent" function class of μ,

$$S_\mu = \{c\mu(x, z) + g(z) : c > 0, g : \mathbb{R}^p \to \mathbb{R}\}.$$

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Good predictive performance $\Longrightarrow$ Good inferential accuracy

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Robustness

Suppose $P_{X|Z}$ unknown, we instead use its estimate $Q^{(n)}_{X|Z}$ to run floodgate.

Theorem (Zhang and Janson (2020))

Under moment conditions on Y and noises, for μ_n with well-behaved moments under both the true distribution and the specified one, we have

$$\mathbb{P}\left(L_n^\alpha(\mu_n) \leq \mathcal{I} + \Delta_n\right) \geq 1 - \alpha - O(n^{-1/2}), \tag{1}$$

where

$$\Delta_n \leq c_1 \sqrt{\mathbb{E}\left[\chi^2\left(P_{X|Z} \,||\, Q^{(n)}_{X|Z}\right)\right]} - c_2\, \mathbb{E}\left[\left(\bar{\mu}_n(X,Z) - \mu^\star(X,Z)\right)^2\right] \tag{2}$$

where $\bar{\mu}_n$ is a particular representative of S_{μ_n} and $\chi^2(\cdot\,||\,\cdot)$ denotes the χ^2 divergence.

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$P_{X|Z}$ is better estimated than $\mathbb{E}[Y \mid X, Z] \Longrightarrow$[2] Floodgate is robust

2. When $\mathcal{I} > 0$

Application to genomic study of platelet count

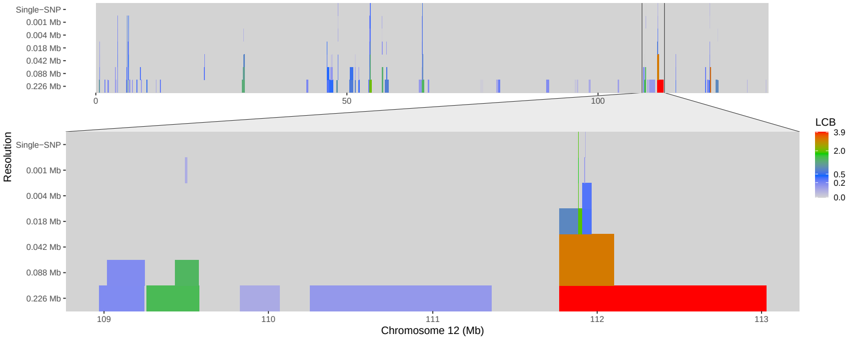

FIGURE – Colored Chicago plot* with the color of each point representing the floodgate LCB for the importance of a group of SNPs on Chromosome 12 in the UK Biobank data at different resolutions (y-axis). Bottom plot shows a zoomed-in region of strong importance.

* Sesia, M., Katsevich, E., Bates, S., Candès, E., & Sabatti, C. (2020). Multi-resolution localization of causal variants across the genome. Nature communications, 11(1), 1-10.

Takeaways

Floodgate : a new inferential approach for variable importance.

- Focus on an interpretable, sensitive and nonparametric measure of variable importance : the mMSE gap.
- Provide valid and robust lower confidence bounds for the mMSE gap.
- Allow flexible regression algorithms, good predictive performance leads to good inferential accuracy.

See more extensions in our paper :

1. Co-sufficient floodgate relaxes the assumptions to only knowing a model for $P_{X|Z}$
2. Floodgate for a different measure of variable importance.
3. Inference on group variable importance.
4. Transporting floodgate inference to a different covariate distribution.
5. Adjusting for multiplicity and selection effects.

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